Joe Dinius
HW 5

MATH 534A
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1. Suppose that M is a smooth manifold. Show that if M is orientable and is covered by *connected* charts $\{U_\alpha, \phi_\alpha\}$ then it is possible to turn $\{U_\alpha, \phi_\alpha\}$ into a covering with consistent orientation by replacing some of the charts ϕ_α with the chart $(-\phi_\alpha^1, \phi_\alpha^2, \ldots, \phi_\alpha^m)$. Use this to show that RP^2 is not orientable and RP^3 is orientable.

   Solution. Let ω be a nowhere vanishing n-form that orients M. On any connected chart, $\{U_\alpha, \phi_\alpha\}$, there is a strictly-positive or strictly-negative function f, by continuity, s.t.

   $$\omega = f d\phi_\alpha^1 \wedge \cdots \wedge d\phi_\alpha^n.$$

   If f is negative, the first coordinate can be changed to ensure consistent orientation of the chart; that is, by inverting the sign on the first coordinate.

   For RP^2, consider the mapping $\pi : S^2 \to \mathrm{RP}^2$, $\pi(p) = \{p, -p\}$. π is a local diffeomorphism. Now, let $r : S^n \to S^n$ be reflection through the origin. Then, $\pi \circ r = \pi$. If RP^2 is orientable, then it is safe to assume that π is orientation preserving, which would mean that $\pi \circ r$ is orientation preserving. This is not possible only when r preserves orientation, which cannot happen in RP^2. Therefore, RP^2 is not orientable.

   For RP^3, consider π and r as above. We can orient each tangent space $T_{[p]}\mathrm{RP}^3$ as follows: choose a representative element $q \in [p] = \{p, -p\}$. Choose a basis of $T_q S^3$ which is in its orientation class, and let the image of this basis under $d\pi$ determine the orientation class of $T_{[p]}\mathrm{RP}^3$. This orientation is well defined because it is not affected by whether $q = p$ or $q = -p$. If (b_1, b_2, b_3) is a basis in the orientation class of $T_p S^3$ and (b_1', b_2', b_3') is a basis in the orientation class of $T_{-p}S^3$, then $(d\pi_p(b_1), d\pi_p(b_2), d\pi_p(b_3))$ and $(d\pi_{-p}(b_1'), d\pi_{-p}(b_2'), d\pi_{-p}(b_3'))$ belong in the same orientation class of $T_{[p]}\mathrm{RP}^3$ since

   $$d\pi_p(b_i) = d(\pi \circ r)_p(b_i) = d\pi_{r(p)} \circ dr_p(b_i) = d\pi_{-p} \circ dr_p(b_i).$$

   and r preserves orientation. Therefore, RP^3 is orientable.

2. Define the 2×2 matrix,

   $$\phi_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

   Show that $\phi_{s+t} = \phi_s \phi_t$ so that ϕ_t defines a one parameter group of diffeomorphisms of $\mathbb{R}^2$. let E_m denote the 1-dimensional submanifold of $\mathbb{R}^2$ defined by,

   $$\{(e, p) | e^2 - p^2 = m^2, e > 0\}$$

   The manifold E_m is the positive energy mass shell for relativistic particles of mass m in 1 space and 1 time dimension (e is the energy and p is the momentum). Show that ϕ_t induces a one parameter family of diffeomorphisms of E_m. This action turns out to be the same as a Lorentz force boost to a moving frame in the physics interpretation. If $m > 0$ show that

$$\mathbb{R} \ni s \to (m \cosh s, m \sinh s) \in E_m$$

is a diffeomorphism of $\mathbb{R}$ with E_m. This map thus defines a coordinate chart on E_m. Show that the one form $\omega = ds$ defined by this coordinate chart is invariant under ϕ_t, that is,

$$\phi_t^* \omega = \omega \ \forall t \in \mathbb{R}.$$

The map $E_m \ni (p, e) \to p$ also defines a global chart, p, on E_m. Find an expression for ω in the chart p. This invariant "volume" form on E_m can be used to construct the Hilbert space of states $L^2(E_m, \omega)$ for particles of spin 0 and mass m in relativistic quantum mechanics.

Solution.

$$\phi_s \phi_t = \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

$$= \begin{pmatrix} \cosh s \cosh t + \sinh s \sinh t & \cosh s \sinh t + \sinh s \cosh t \\ \sinh s \cosh t + \cosh s \sinh t & \cosh s \cosh t + \sinh s \sinh t \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(s+t) & \sinh(s+t) \\ \sinh(s+t) & \cosh(s+t) \end{pmatrix} = \phi_{s+t}$$

I next need to show that action of ϕ_t on a point $(e, p) \in E_m$ is in E_m again. That is, take $(e, p) \in E_m$, then

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} e \\ p \end{pmatrix} = \begin{pmatrix} e \cosh t + p \sinh t \\ e \sinh t + p \cosh t \end{pmatrix}$$

$$\begin{aligned} (e \cosh t + p \sinh t)^2 - (e \sinh t + p \cosh t)^2 &= (e^2 \cosh^2 t + 2pe \cosh t \sinh t + p^2 \sinh^2 t) \\ &- (e^2 \sinh^2 t + 2pe \cosh t \sinh t + p^2 \cosh^2 t) \\ &= (e^2 - p^2)(\cosh^2 t - \sinh^2 t) \\ &= e^2 - p^2 \\ &= m^2. \end{aligned}$$

So, $\phi_t (e \ p)^T \in E_m$. Since E_m is a regular submanifold of $\mathbb{R}^2$ and inclusion map from E_m to $\mathbb{R}^2$ is smooth, and ϕ_t is smooth, the composition $i \circ \phi_t$ is smooth. This shows that ϕ_t generates a one parameter family of diffeomorphisms of E_m to itself.

Now, I need to show that $s \to (m \cosh s, m \sinh s)$ is a diffeomorphism of $\mathbb{R}$ with E_m. It has an inverse which is a function, therefore it is one-to-one and onto; cosh and sinh are C^∞ and the inverse function theorem implies that the map has C^∞ inverse. Therefore, the map is a diffeomorphism.

Consider the coordinates s on E_m, and r on $\phi_t(E_m) = E_m$.

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} m \sinh s \\ m \cosh s \end{pmatrix}$$

$$= \begin{pmatrix} m(\sinh s \cosh t + \sinh t \cosh t) \\ m(\sinh s \sinh t + \cosh s \cosh t) \end{pmatrix}$$

$$= \begin{pmatrix} m \sinh(s+t) \\ m \cosh(s+t) \end{pmatrix}$$

So, the coordinate r is just $s+t$, and therefore

$$\phi_t^*(dr) = d(s+t) = ds + 0 = ds$$

So, ω is invariant under action by ϕ_t.

To express, ω in terms of the p-coordinate, consider

$$p = m \sinh s$$

Implicit differentiation yields

$$\begin{aligned} dp &= m \cosh s ds \\ &= e ds \end{aligned}$$

3. Suppose that f is a smooth function on $\mathbb{R}^n$ and $df_p \neq 0$ when $f(p) = 0$, so that $M = \{p | f(p) = 0\}$ is a regular submanifold of $\mathbb{R}^n$. Define a unit normal field on M by,

$$n(p) = \frac{\nabla f_p}{|\nabla f_p|},$$

where $|\cdot|$ is the usual Euclidean length and ∇f is the gradient of f on $\mathbb{R}^n$. Define an $n-1$ form, Ω, on M by contracting $n(p)$ with the standard volume form on $\mathbb{R}^n$. That is, define,

$$\Omega_p(v_1, \ldots, v_{n-1}) = dx^1 \wedge \cdots \wedge dx^n(n(p), v_1, \ldots, v_{n-1}), \text{ where } v_j \in T_pM$$

Show that Ω is a non-vanishing volume form on M so that M is always orientable. Find the volume form Ω on S^{n-1} and show that it is invariant under the action of $g \in \mathrm{SO}(n)$ on S^{n-1}. Hint: express Ω directly in terms of the differentials dx^j (it makes perfectly good sense to restrict these to the tangent space T_pS^{n-1}). To say that Ω is invariant under the action of g is just to say $g^*\Omega = \Omega$.

Solution. Since $n(p)$ is normal to elements in the tangent space T_pM, the form Ω_p represents the (signed) volume of M and hence must be nowhere vanishing on M. Therefore, M is always orientable.

Let Ω be defined as follows

$$\Omega = \det(n(p), v_1, \ldots, v_{n-1}).$$

As before, since the v_j are orthogonal to $n(p)$, this is indeed a volume form on S^{n-1}.

Now, I need to show that this is invariant under action by $g \in \mathrm{SO}(n)$.

$$g^*\Omega_p(v_1, \ldots, v_{n-1}) = \Omega_{g(p)}(dg_p(v_1), \ldots, dg_p(v_{n-1}))$$

Since g has determinant 1, g acting on Ω will not change the determinant and hence leaves Ω invariant.